**Integrating over the random effects in a Bayesian hierarchical model**

(first draft) by Jesse

**I. Some lemmas**

*Completing the square in matrix form*

If is symmetric and invertible,

,

or equivalently,

**.**

*Proof*

Starting with the right hand side,

and simplifying and using the symmetry of , we have

. ■

*Proposition*

If random vector has density ,

where ,

then .

*Proof*

Suppose for some constant .

Then .

The quantity is a multivariate normal density,

so integrating both sides wrt gives ,

so , so ,

which is the density. ■

*Proposition*

If ,

where ,

then .

Proof

Rewrite as , so that we have

,

and the conclusion follows by the previous result. ■

*Proposition*

If ,

where , and is pd,

then .

*Proof*

Suppose ,

and also for some .

Then , so .

Also , so , so .

Therefore by the previous proposition, . ■

**II. Model specification in matrix form**

For and , is subject, is timepoint, is outcome.

In this model we have timepoints and 3 outcomes.

For and , is subject, is index.

Ignore the effect of diagnostic group for now.

The matrix equation for subject is , or

,

with , and with ,

where is the covariance matrix of the outcomes,

and with .

For compactness, we might write

.

**III. Integrating out**

Assume are known. To integrate out , use the relationship

.

The densities are

and ,

where (dropping subscripts)

so the joint density is

where .

We add the quadratic forms and refactor as follows:

.

Let , and let ,

noting that depends on , and we have

.

Now if we integrate over , the kernel

becomes 1, and we have

.

At this point, WLOG we let , remembering to add it to the mean of at the end:

.

Let ,

and let.

Then we have ,

and so .

Adding back , we finally have

. ■

Quick sanity check: do we see a anywhere? No.

Another one: do dimensions match?

is , is ), so is

is , is , so and are

is , so is

so they do match.

**IV. Another method**

This time we make the means of the random intercept and slope fixed, part of ,

so that each has mean .

The matrix equation for subject is , or

~~,~~

with , and with ,

where is the covariance matrix of the outcomes,

and with .

Assume are known. To integrate out , use the relationship

.

The densities are

and ,

where, dropping subscripts and letting , we have

so the joint density is

where .

Letting and , we have

.

Now if we integrate over , the kernel

becomes 1, and we have

which means that ,

and so ,

where **,**

and using the Woodbury matrix identity,

.

Woodbury matrix identity, Wikipedia:

So for us

,

\* 2449

Suppose subject has timepoints.

,

what is the variance of

?

If subject i has 3 timepoints, our model fit (bivariate model) is specified as (suppress i)

We want to find

We need permutation matrix that does .